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CAPACITARY INEQUALITIES FOR ENERGY

BY
MURALI RAO
Department of Mathematics, University of Florida, Gainesville, Florida, USA

ABSTRACT
A strong capacitary inequality for not necessarily symmetric Markov processes is obtained.

§1. Introduction

Consider a bounded domain D in R^d, the Euclidean d-dimensional space, and let G denote the Green function of D (Doob [D1] or M. Rao [R1]). For a signed measure m denote by Gm its potential, and the energy norm of Gm by $\int Gm(x)m(dx) = \| Gm \|_e^2$ (assuming as usual that the corresponding quantities for $|m|$ are finite). We have

$$\| Gm \|_e^2 = \int Gm(x)m(dx) = \int |\nabla Gm|^2(x)dx, \tag{1.1}$$

∇ denoting gradient operator. The capacity $C(A)$ of a Borel set A is defined as

$$C(A) = \sup\{m(A)\}, \tag{1.2}$$

where the supremum is taken over all non-negative measures m concentrated on A whose potentials are $\leqq 1$ everywhere on D. With these definitions the following capacitary estimate of energy is known (Adams [A1], Mazya [M1], Dahlberg [DA1]):

If m is a positive measure and $u = Gm$ then

$$\int_0^\infty tC(u > t)dt \leqq \| u \|_e^2. \tag{1.3}$$

Estimates of the type (1.3) have found applications in trace theorems. For

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example, using these one can show that for a domain D with nice boundary, restrictions of elements of the Sobolev space H^1 to the boundary may be defined. The notion of energy can be extended to very general Markov processes [ZPMR1] with not necessarily symmetric potential kernels. In this note we modify (1.3) to this situation. It will be noticed that the proofs in this more general case are much simpler than the published proofs in the classical case.

§2. Notation and terminology will generally be that of [ZPMR1]. Thus X will denote a transient Hunt process on a l.c.c. state space E with life time ζ. We assume that there is an excessive reference measure ξ also denoted dx and a potential kernel $u = u(x, y)$ such that for all nonnegative Borel functions f

$$Uf(x) = E^x\left[\int_0^\infty f(X_t)dt\right] = \int u(x, y)f(y)dy. \tag{2.1}$$

Assume X has Borel transition semigroup which we denote by P_t. A Borel function $s \geqq 0$ is called *excessive* if it is finite a.e. relative to dx and

$$s = \sup_{t>0} P_t s. \tag{2.2}$$

An excessive function s is called *purely excessive* if $\lim_{t\to\infty} P_t s = 0$ a.e. and it is called a *class* (D) *potential* if $P_{T_n} s$ decreases to zero a.e. whenever the sequence of stopping times T_n increases to ∞.

We now recall briefly the definition and some results on energy. Details may be found in [ZPMR1], [GMR1], and [R3].

With each purely excessive function s we can associate a number $L(s)$, $0 \leqq L(s) \leqq \infty$, called the *mass functional* of s as follows:

$$L(s) = \sup_{t>0} t^{-1} \int (s - P_t s)(x)dx. \tag{2.3}$$

Let us recall that $L(s) = \sup_\lambda \lambda(s)$ for meaures λ with $\lambda U \leqq \xi$. L is monotone in s and continuous along increasing sequences. This allows the extension of L to all excessive functions by monotonicity.

Now we recall the definition of energy of a *class*(D) *potential* as given in [ZPMR1]. Every class(D) potential s is generated by a natural additive functional A defined off a polar set. If $p = E[A_\infty^2]$ is finite a.e. then p itself is necessarily a class(D) potential. In particular it is purely excessive. If $L(p) < \infty$ we say that s has *finite energy* and put

(2.4) $$\| s \|_e^2 = L(p).$$

If r and s are class(D) potentials of finite energy generated by natural additive functionals A and B, respectively, their mutual energy $(r, s)_e$ is defined by

(2.5) $$(r, s)_e = LE^{\cdot}[A_\infty B_\infty].$$

The above definition extends in a natural way to the linear space of differences of class(D) potentials of finite energy. And this space becomes a pre-Hilbert space, except for the fact that there may be some nonzero elements of zero energy. For details see [ZPMR1]. However, we shall talk as if energy were a true norm. If s is the potential of a function f, then

(2.6) $$\| s \|_e^2 = 2 \int fUf dx$$

which is the classical definition of energy except for the factor 2.

For any Borel set A, $P_A 1$ is excessive and we define its capcity $C(A)$ by

(2.7) $$C(A) = L(P_A 1)$$

where $P_A 1(x) = P^x[T_A < \infty]$, T_A being the first hitting time to A.

We find it easier to prove our results for potentials of functions. To pass to the general case we use the following theorem. This theorem is a simple corollary of Theorem 1.4 of [ZPMR1].

THEOREM 2.1. *If s is a class(D) potential of finite energy, there is a sequence $s_n = Uf_n$ of potentials of functions increasing to s and converging to s in energy [(i.e., $\| s - s_n \|_e$ tends to zero).*

We record one more simple fact: If s is excessive, then for each $\lambda > 0$

(2.8) $$\lambda C(s > \lambda) \leqq Ls.$$

This is simple because $\lambda P_{(s>\lambda)} 1 \leqq s$, now apply L and use monotonicity of L. Our first capacitary estimate is the following.

LEMMA 2.2. *Let s be a class(D) potential of finite energy. Then*

(2.9) $$\int_0^\infty \lambda C(s > \lambda) d\lambda \leqq 2 \| s \|_e^2.$$

PROOF. We prove this theorem assuming $s = Uf$ and then appeal to Theorem 2.1. For any $\lambda > 0$, let $A(\lambda)$ and $B(\lambda)$ denote the sets $(s \leqq \lambda)$ and $(s > \lambda)$ respectively. Let us also use the same notation for a set and its indicator function. Then

$$s = U[fA(\lambda)] + U[fB(\lambda)]. \tag{2.10}$$

Now $U[fA(\lambda)] \leqq Uf = s \leqq \lambda$ on the set $A(\lambda)$ which contains the set where $fA(\lambda) > 0$. By the maximum principle this holds everywhere. Hence, from (2.10),

$$s \leqq \lambda + U[fB(\lambda)]. \tag{2.11}$$

In particular, the set $(s > 2\lambda) \subset (U[fB(\lambda)] > \lambda)$ so that

$$\lambda C(s > 2\lambda) \leqq \lambda C[U[fB(\lambda)] > \lambda] \leqq LU[fB(\lambda)] \leqq \int f(x)B(\lambda)(x)dx.$$

Integrate both sides of this last inequality relative to λ from 0 to ∞ to get

$$\int_0^\infty \lambda C(s > \lambda)d\lambda \leqq 4 \int f.\, sdx = 2 \, \| s \|_e^2.$$

The proof is complete.

Our next result is an estimate in the other direction.

LEMMA 2.3. *Let s be excessive of class(D). Then*

$$2^{-1} \| s \|_e^2 \leqq \int_0^\infty \lambda C(s > \lambda)d\lambda. \tag{2.12}$$

PROOF. Again we prove the result assuming $s = Uf$ and then appeal to Theorem 2.1. Let $B(\lambda)$ be as in the proof of Lemma 2.2. Supose $s(x) = 2\lambda_0$. Then $s(x) > \lambda$ for each $\lambda_0 < \lambda < 2\lambda_0$ so that

$$x \in B(\lambda), \qquad \lambda_0 < \lambda < 2\lambda_0. \tag{2.13}$$

Since $B(\lambda)$ is finely open, $P_{B(\lambda)}1(y) = 1$ for $y \in B(\lambda)$. From (2.13)

$$\int_{\lambda_0}^\infty P_{B(\lambda)}1(x)d\lambda \geqq \int_{\lambda_0}^{2\lambda_0} P_{B(\lambda)}1(x)d\lambda = \lambda_0 = 2^{-1}s(x).$$

It follows that if $s(x) > 2u$ we have

$$\int_u^\infty P_{B(\lambda)}1(x)d\lambda \geqq \int_{2^{-1}s(x)}^\infty P_{B(\lambda)}1(x)d\lambda \geqq 2^{-1}s(x),$$

so that on the set $B(2u)$

$$U[fB(2u)](x) \leqq Uf(x) = s(x) \leqq 2 \int_u^\infty P_{B(\lambda)}1(x)d\lambda \tag{2.14}$$

and hence, by the maximum principle, everywhere. Applying L to (2.14)

$$\int fB(2u)dx \leqq 2\int_u^\infty C(B(\lambda)d\lambda.$$

Integrate both sides of the above inequality relative to u and use Fubini to finish the proof.

Let us record the results of the above two lemmas as

THEOREM 2.4. *Let s be a class(D) potential of finite energy. Then*

$$2^{-1}\|s\|_e^2 \leqq \int_0^\infty \lambda C(s>\lambda)d\lambda \leqq 2\|s\|_e^2. \tag{2.15}$$

Another approach to Theorem 2.4 due to Professor P. A. Meyer is as follows: Since $L(s)=\sup \mu(s)$ with $\mu U \leqq dx$ it suffices to prove that

$$\left\langle \mu, \int_0^\infty \lambda P_{(s>\lambda)}1\,d\lambda\right\rangle \leqq 2\langle \mu, E^{\cdot}[A_\infty^2]\rangle.$$

Now $P_{(s>\lambda)}1 = P^{\cdot}(s^* > \lambda)$ where s^* is the r.v. $\sup_t s(X_t)$, so the left side of the last inequality is just $2^{-1}E^\mu[s^{*2}]$. Thus (2.15) is a particular case of Doob's inequality. However, the proof given uses only the weak maximum principle. Even linearity is not used. Cases in point are Riesz potentials, potentials arising from Schrödinger operators, maximal operators, etc. The interested reader may consult a preprint coming out in Aarhus University, Denmark.

Next we deal with a difference of class(D) potentials of finite energy. But before we do this let us give some results on maximal functions along paths.

Let s be a class(D) potential. Put

$$s^* = \sup_{t>0} s(X_t).$$

Then $E^{\cdot}[(s^{*2})]$ is also excessive provided it is finite a.e. And

$$4^{-1}\|s\|_e^2 \leqq LE^{\cdot}[s^{*2}] \leqq 2\|s\|_e^2. \tag{2.16}$$

To see this write

$$s(X_t) = E^x[A_\infty/F_t] - A_t = M_t - A_t$$

valid for all x such that $s(x) < \infty$. Then

$$E^x[(s^*)^2] \leqq E^x[(M^*)^2] \leqq 2E^x[A_\infty^2],$$

the last by Doob's inequality. The right side of (2.16) is thus proved. On the other hand, use Meyer's energy formula:

$$E[A_\infty^2] = E\left[\int_0^\infty \{s(X_t) + s(X_t)_-\} dA_t\right]$$
$$\leqq E[2s^* A_\infty]$$
$$\leqq 2\{E[(s^*)^2] E[A_\infty^2]\}^{2^{-1}}$$

to get $E^{\cdot}[A_\infty^2] \leqq 4E^{\cdot}[(s^*)^2]$. And now apply L.

Now let F be any measurable function and put

$$F^* = \sup |F(X_t)|.$$

Now $\{F^* > \lambda\} = \bigcup_n R_n(B)$ where $B = \{x : F(x) > \lambda\}$. So F^* is measurable by Lemma 10.14, p. 56 of [BG1].

Let, for $t > 0$, T_t be the hitting time to the set $\{|F| > t\}$. Then

$$P^x[F^* > \lambda] = P^x[T_\lambda < \infty].$$

Multiply the above by λ and integrate from 0 to ∞ to get

$$2^{-1} E^x[(F^*)^2] = \int_0^\infty \lambda P^x[T_\lambda < \infty] d\lambda. \tag{2.17}$$

Applying the mass functional L we get

$$\begin{aligned} 2^{-1} L E^{\cdot}[(F^*)^2] &\leqq \int_0^\infty \lambda L P^{\cdot}[T_\lambda < \infty] d\lambda \\ &= \int_0^\infty \lambda C(|F| > \lambda) d\lambda. \end{aligned} \tag{2.18}$$

If $g(x) = E^x\{F^*\}$ then

$$g(X_t) = E^{X_t}[F^*] = E[F^*(\theta_t)/F_t] \leqq E[F^*/F_t]$$

so that by Doob's inequality

$$E^x[(g^*)^2] \leqq 2E^x[(F^*)^2]. \tag{2.19}$$

Combining (2.18) and (2.19):

$$L E^{\cdot}[(g^*)^2] \leqq 4 \int_0^\infty \lambda C[|F| > \lambda] d\lambda. \tag{2.20}$$

By Corollary (1.16), p. 202 of [BG1], except perhaps for a semipolar set of x, $F^* > |F(x)|$ a.s. P^x. Therefore except for a semipolar set $g > |F|$. Combining (2.20) with (2.16) (replace s by g) we get the following theorem:

THEOREM 2.5. *Let F be Borel measurable. Put $g(x) = E^x[F^*]$. Then g is excessive, $g \geqq |F|$ except for a semipolar set and*

$$\|g\|_e^2 \leqq 16 \int_0^\infty \lambda C(|F| > \lambda) d\lambda. \tag{2.21}$$

Further, if $|F|$ is finely lower semi-continuous we also have

$$\int_0^\infty \lambda C(|F| > \lambda) d\lambda \leqq 2 \|g\|_e^2. \tag{2.22}$$

PROOF. We need only show (2.22). If $|F|$ is finely lower semi-continuous we necessarily have $|F| \leqq E[F^*] = g$, so that for each $\lambda > 0$

$$C(|F| > \lambda) \leqq C(g > \lambda).$$

Since g is excessive, Lemma 2.2 applies. That completes the proof.

Now we proceed to the final theorem of this section. For the result below we assume that we have two Hunt processes in duality. Recall that a class(D) potential is called *regular* if the corresponding additive functional is continuous.

The following observation will be helpful in the next result.

OBSERVATION. Let μ be a non-negative measure whose potential $U\mu$ has finite energy. We claim the co-potential $\hat{U}\mu$ of μ is also a class(D) co-potential of finite energy. Let g be a strictly positive bounded and integrable function such that Ug is bounded — the existence of such a function is one of several equivalent definitions of transience. We have

$$\langle \hat{U}\mu, g \rangle = \langle \mu, Ug \rangle \leqq \langle \mu, Ug \rangle + \langle U\mu, g \rangle \leqq \|U\mu\|_e \|Ug\|_e,$$

showing that $\hat{U}\mu$ is finite a.e. Also $\hat{U}\mu$ is purely excessive because $\langle \hat{P}_t \hat{U}\mu, g \rangle = \langle \mu, P_t Ug \rangle$ and this last integral tends to zero by the dominated convergence theorem. Let now Uf_n be a sequence of potentials increasing to $U\mu$. Then $\|\hat{U}f_n\|_e = \|Uf_n\|_e \leqq 2 \|U\mu\|_e$. By Proposition 2.2 of [ZPMR1] $\hat{U}f_n$ converges in measure to $\hat{U}\mu$. By Proposition 0.4 of [ZPMR1] the claim follows.

THEOREM 2.6. *Let $0 \leqq Um$ be a difference of class(D) potentials of finite* *energy. Assume also that $\hat{U}m$ is finely lower semi-continuous. Put*

$$g = E^{\cdot}[(\hat{U}m)^{+*}]. \tag{2.23}$$

Then $g \geqq \hat{U}m$, g is excessive of finite energy and

$$2^{-1}\int_0^\infty \lambda C[\hat{U}m > \lambda]d\lambda \leqq \| g \|_e^2 \leqq 16 \| Um \|_e^2. \tag{2.24}$$

PROOF. Um has finite energy if and only if $U|m|$ also has finite energy. By the dual version of Lemma 2.2

$$\int_0^\infty \lambda \hat{C}[\hat{U}|m| > \lambda]d\lambda < \infty$$

and in particular

$$\int_0^\infty \lambda \hat{C}[(\hat{U}m)^+ > \lambda]d\lambda < \infty.$$

Since capacity and co-capacity of a set are equal, the last inequality can be written

$$\int_0^\infty \lambda C[(\hat{U}m)^+ > \lambda]d\lambda < \infty. \tag{2.25}$$

By Theorem 2.5

$$g = E[(\hat{U}m)^{+*}] \tag{2.26}$$

is excessive, of finite energy and dominates $(\hat{U}m)^+$. If μ is the Revuz measure of g we have

$$(g, Um)_e \geqq \int g dm. \tag{2.27}$$

because by assumption $Um \geqq 0$. To see this let Uh_n increase to g. Then

$$(Uh_n, Um)_e = (h_n, Um) + \int Uh_n dm \geqq \int Uh_n dm.$$

Now appeal to Theorem 1.4 of [ZPMR1].

Now we may write g in the following way:

$$\begin{aligned} g(x) &= E^x[(\hat{U}m)^{+*}] \\ &= \int_0^\infty P^x[(\hat{U}m)^{+*} > \lambda]d\lambda \\ &= \int_0^\infty P^x[T_\lambda < \infty]d\lambda \end{aligned}$$

where T_λ is the hitting time to the set

$$O_\lambda = \{(\hat{U}m)^+ > \lambda\}. \tag{2.28}$$

Thus

$$\int g dm = \int_0^\infty d\lambda \int P^\cdot[T_\lambda < \infty]dm. \tag{2.29}$$

Now we estimate the integral on the right side. For each λ, O_λ defined in (2.28) is finely open. Therefore there is a sequence f_n of functions vanishing off O_λ such that Uf_n increases to $P[T_\lambda < \infty]$. We get

$$\begin{aligned} \int P[T_\lambda < \infty]dm &= \lim_n \int Uf_n dm = \lim_n \int f_n \hat{U}m dx \\ &\geqq \lim_n \lambda \int f_n dx = \lambda C\{(\hat{U}m)^+ > \lambda\} \end{aligned}$$

because on the set $(f_n > 0)$, $\hat{U}m = (\hat{U}m)^+ > \lambda$. Using (2.27), (2.29) and (2.30) we can thus estimate

$$(g, Um)_e \geqq \int g dm \geqq \int_0^\infty \lambda C\{(\hat{U}m)^+ > \lambda\}d\lambda \geqq (16)^{-1} \| g \|_e^2, \tag{2.31}$$

the last inequality following from (2.21). Since $(g, Um)_e \leqq \| Um \|_e \| g \|_e$ we obtain $\| g \|_e \leqq 4 \| Um \|_e$. That completes the proof.

Example. The following example shows why we have difficulty in improving the above result. Consider the uniform motion to the right on $[0, \infty)$, the reference measure being the Lebesgue measure. The potential Uf of $f = (t-1)e^{-t}$ is te^{-t}: $Uf = te^{-t}$. $Uf \geqq 0$ and has energy zero. $\hat{U}f = -te^{-t}$. Any excessive function whose energy is dominated by the energy of Uf must have zero energy and hence must vanish. Thus we are unable to say $g \geqq |\hat{U}m|$ in Theorem 2.2.

Comment. From Theorem 2.5 and the remarks following Theorem 2.4 we see that if F is finely lower semi-continuous, then there is an excessive potential

s of finite energy dominating $|F|$ if and only if the right side of (2.21) is finite. Indeed if s has finite energy and $s \geqq |F|$, then for each $t > 0$, $C(|F| > t) \leqq C(s > t)$ and (2.15) can be used to deduce the finitenss of the right side of (2.21).

Using this comment we have the following corollary of Theorem 2.6.

COROLLARY 2.7. *Let Um and $\hat{U}m$ be as in Theorem 2.6. Then there is a co-excessive function h such that $h \geqq \hat{U}m$ and $\| h \|_e \leqq 16 \| Um \|_e$. In particular, if both Um and $\hat{U}m$ are non-negative there is an excessive function s such that $s \geqq Um$ and $\| s \|_e \leqq 16 \| Um \|_e$.*

Let us explain our interest in the above results. Let X be the Brownian motion process in d-dimensions killed upon exit from a bounded domain. The completion, in energy norm, of the space of differences of class(D) potentials of finite energy is called the Sobolev space H^1. The modulus contraction operates on this space:

$$f \in H_0^1 \quad \text{implies} \quad |f| \in H_0^1.$$

This fact is sometimes used to show Hypothesis (H) of Hunt:

Every excessive function is regular.

In the more general non-symmetric situation we have the following result.

THEOREM 2.8. *Suppose there is a constant α with the property:*

If s is a difference of class(D) potentials of finite energy there exists a class(D) potential p such that

$$|s| \leqq p \quad \textit{and} \quad \| p \|_e \leqq \alpha \| s \|_e.$$

Then Hypothesis (H) *of Hunt holds.*

PROOF. Let s be a class(D) potential of finite energy. There exists a sequence $\{s_n\}$ of potentials of the form $s_n = Uf_n$ increasing to s and converging to s in energy. For large n, $\| s - s_n \|_e$ is therefore small. By hypothesis there is a class(D) potential p_n of small energy such that $|s - s_n| \leqq p_n$. Let A, A_n denote the additive functional of s, s_n respectively. Then

$$E^{\cdot}\left[\int (s(X_t)_- - s(X_t))dA_t\right] \leqq E^{\cdot}\left[\int (p_n(X_t)_- + p_n(X_t))dA_t\right]$$

$$\leqq E^{\cdot}[B_{n,\infty} A_\infty]$$

where B_n is the additive functional generating p_n. Applying the mass functional L we obtain

$$L\left[E^{\cdot}\left(\int (s(X_t)_- - s(X_t))dA_t\right)\right] \leqq L[E^{\cdot}(B_{n,\infty}A_\infty)] = (p_n, s)_e \leqq \| p_n \|_e \| s \|_e.$$

Since the right side above tends to zero as n tends to infinity, the excessive function $E^{\cdot}(\int(s(X_t)_- - s(X_t))dA_t)$ must be zero. In other words s is regular. That completes the proof.

§3. Further results

Let us retain the notation and terminology of §2. Let us make the assumption that the set of excessive potentials of finite energy is complete in energy norm. General conditions guaranteeing this are given in [GMR2]. The following is a generalisation of the well-known equilibrium principle.

THEOREM 3.1. *Let $f = Um$ be of finite energy. Then there exists a potential $p = U\mu$ of finite energy such that*

$$\text{(3.1)} \qquad \begin{cases} (1)\ (f, p)_e = \| p \|_e^2, \text{ in particular } \| p \|_e \leqq \| f \|_e; \\ (2)\ p + \hat{p} \geqq f + \hat{f}; \\ (3)\ p + \hat{p} = f + \hat{f}\ \mu\text{-a.e. provided } f \text{ is regular.} \end{cases}$$

Here, if $s = Um$ we put $\hat{s} = \hat{U}m$.

PROOF. The set of excessive potentials of finite energy is convex, and complete by assumption. By elementary Hilbert space theory there is a unique potential $p = U\mu$ minimizing

$$\text{(3.2)} \qquad \| f - p \|_e = \inf \| f - h \|_e$$

where the inf is over all potentials h of finite energy. Replace h by $p + th$ and expand to get

$$2(f - p, h)_e \leqq t^2 \| h \|_e^2,$$

which implies as $t \to 0$

$$\text{(3.3)} \qquad (f - p, h)_e \leqq 0.$$

Taking h of the form $h = U\phi$, $\phi \geqq 0$, we get from (3.3) for all $\phi \geqq 0$ such that $U\phi$ has finite energy

$$\int \{U(m-\mu)+\hat{U}(m-\mu)\}\phi(x)dx \leqq 0.$$

In other words we must have

$$[U+\hat{U}](m-\mu)\leqq 0.$$

This is (2) of (3.1). To complete the proof we write, for any Borel set A, $p_A = U\mu A$. Recall that a set and its indicator are denoted by the same letter. Then $p + tp_A$ is excessive for $-1 \leqq t \leqq 1$. Similarly as above one gets

$$(f-p,\, p_A)_e = 0. \tag{3.4}$$

This is (1) of (3.1) with $p = p_A$. Finally, if f is regular the inner product in (3.4) is just

$$\int [U+\hat{U}](m-\mu)(x)\mu(dx)=0$$

and we get (3) of (3.1) That completes the proof.

One class of examples where the hypothesis of Theorem 3.1 is satisfied is given below.

Example. Let X be the Levy process on R^d with exponent ψ:

$$E\{\exp i(\alpha, X_t)\} = \exp\{-t\psi(\alpha)\}.$$

Let us assume that the transition semigroup of X has bounded and continuous densities which vanish at ∞. It is shown in [R4] that a difference s of 1-excessive class(D) potentials of finite energy is square summable and its 1-energy is given by

$$\| s \|_{e,1} = \int (1+\operatorname{Re}\psi)|\tilde{s}|^2$$

where $\tilde{s}$ is the Fourier transform of s. Using this expression for energy it is easy to show that the class of excessive potentials of finite energy is complete in energy norm. Indeed, let $\{s_n\}$ be a sequence of excessive potentials which is Cauchy in energy norm. Since the energy norm dominates the L^2 norm, this sequence is also Cauchy in L^2. By choosing a subsequence if necessary let us assume that s_n converges a.e. and in L^2 to an excessive function s. A result in [ZP1] implies that if there exists one potential vanishing at ∞ (in our case there are many), s must necessarily be a class(D) potential of finite energy. Now

$$\| s_n - s \|_e^2 \leqq \liminf_m \int (1 + \text{Re}\,\psi) | \tilde{s}_n - \tilde{s}_m |^2$$

is small if n is large, completing the proof.

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